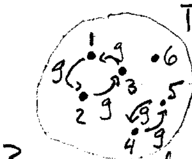
Math 122 Monday, November 7

$S_n = \mathrm{Aut}\{1,2,\dots,n\}$ finite group of order $n!$ $g \in S_n$ acts on $T = \{1,2,\dots,n\}$ Consider $\langle g \rangle = H$ orbits on T.

<u>Note</u>: orbits of $\langle g \rangle$ give cycle decomposition of g  $\longleftrightarrow g = (123)(45)(6)$

<u>defn</u> the support of g, $\mathrm{Supp}(g) = T - \{\text{fixed points}\}$ = subset of T which is moved by g.
ex. $\mathrm{Supp}(e) = \phi$, $\mathrm{Supp}((ab)) = \{a,b\}$, $\mathrm{Supp}(g) = \{a,b,c\} \Rightarrow g = (abc)$ or $g = (acb)$.

<u>Prop</u> If g and h have disjoint support then $gh = hg$ commute in S_n.
Pf: Must check $gh(i) = hg(i)$ for all $i \in T$. If i is fixed by both g and h then $gh(i) = hg(i) = i$. If $i \in \mathrm{Supp}(h)$ then i is fixed by g and $h(g(i)) = h(i) = g(h(i))$ where the last step is true because $i \in \mathrm{Supp}(h) \Rightarrow h(i) \in \mathrm{Supp}(h) \Rightarrow h(i)$ is fixed by g. The third case ($i \in \mathrm{Supp}(g) \Rightarrow i$ fixed by h) is analogous.

<u>Aside</u> Say $i \in \mathrm{Supp}(h)$ but $h(i)$ is not $\Rightarrow h(h(i)) = h(i) \Rightarrow h^{-1}(h(h(i))) = h^{-1}h(i) \Rightarrow i = h(i) \Rightarrow i \notin \mathrm{Supp}(h) \Rightarrow \Leftarrow$.

In general, we can write any $g \in S_n$ as the product of disjoint cycles (orbits for $\langle g \rangle$) $g = g_1 \cdots g_k$ where $g_1 = (i_1 i_2 \cdots i_m)$ etc. The order of g_1 is $S_n = m$, its length. (factors commute so order doesn't matter)
So the <u>order</u> of g = lcm of the lengths of the disjoint cycles.

ex. $g = (12)(345)(6) = (6)(345)(12) = (453)(21)(6)$ etc $g^m = (12)^m (345)^m (6)^m = e$ iff $6 \mid m$

Recall the sign homomorphism $S_n \to \langle \pm 1 \rangle$. So $\mathrm{sign}(g) = \prod \mathrm{sign}(g_i)$. $(i_1 i_2 \cdots i_m) = (i_1 i_m)(i_1 i_{m-1}) \cdots (i_1 i_2)$ the product of $m-1$ transpositions. $\mathrm{Sign}((ab)) = -1 \Rightarrow \mathrm{Sign}((i_1 i_2 \cdots i_m)) = (-1)^{m-1}$. e.g. $\mathrm{Sign}((12345)(678)(9\,10)(11\,12)) = (-1)^4(-1)^2(-1)(-1) = 1$. $(12345)(678)(9\,10)(11\,12)$ has order $5 \times 3 \times 2 = 30$.

Note also we've shown that every $g \in S_n$ is a product of transpositions. Also true (though less obvious) that every $g \in A_n$ is a product of 3-cycles.

In fact, can show that S_n is generated by $\{(12), (23), \dots, (n-1\ n)\}$ the set of $n-1$ <u>adjacent transpositions</u>.
If $g = (i\ i{+}1)$ $h = (j\ j{+}1)$ has disjoint support, they commute. Otherwise $ghg = hgh$. (Always $g^2 = h^2 = e$.)

<u>defn</u> In a group the <u>commutator</u> $[g,h] = ghg^{-1}h^{-1}$. Note $g,h \in S_n \Rightarrow [g,h] \in A_n$ as $\mathrm{sign}(g) = \mathrm{sign}(g^{-1})$.

<u>Prop</u> 1) If $\mathrm{Supp}(g) \cap \mathrm{Supp}(h) = \phi$ then $[g,h] = e$.
2) If $\mathrm{Supp}(g) \cap \mathrm{Supp}(h) = \{i\}$ then $[g,h] = (i\ g(i)\ h(i))$ a 3-cycle.
Pf 1) By previous prop, $[g,h] = gg^{-1}hh^{-1} = e$.
Pf 2) As $i \in \mathrm{Supp}(h)$, $h^{-1}(i) \neq i \in \mathrm{Supp}(h) \Rightarrow h^{-1}(i)$ is fixed by g. So $hg^{-1}h^{-1}(i) = i$ $\Rightarrow [g,h](i) = g(i)$. Similarly $g(i)$ is fixed by $h \Rightarrow g^{-1}hg(i) = i$ and $h(i)$ is fixed by $g \Rightarrow [g,h](g(i)) = h(i)$. Finally $[g,h](h(i)) = ghg^{-1}(i) = i$ as before.
It remains to show that $[g,h]$ fixes $j \notin \{i, g(i), h(i)\}$. As $j \neq i$ j is either fixed by h or fixed by g (or both, in which case this is obvious). In the first case, h also fixes $g^{-1}(j)$ (otherwise $g^{-1}(j) \in \mathrm{Supp}(g) \cap \mathrm{Supp}(h) \Rightarrow j = g(i)$) so $[g,h]j = ghg^{-1}(j) = gg^{-1}(j) = j$. The other case is similar.

Claim In S_n $g(i_1 \cdots i_m)g^{-1} = (g(i_1) \cdots g(i_m))$, a cycle of the same length.
Pf: Clear that when you apply this to $g(i_k)$ you get $g(i_{k+1})$. Say $j \neq g(i_k) \Rightarrow g^{-1}(j) \neq i_k$ for any k. Then $g^{-1}(j)$ is fixed by the cycle $\Rightarrow j$ is fixed by $g(i_1 \cdots i_m)g^{-1}$

More generally, if $h = (\cdots)\cdots(\cdots)$ is a product of disjoint cycles
$ghg^{-1} = g(\cdots)g^{-1}\, g(\cdots)g^{-1} \cdots g(\cdots)g^{-1}$. So by the above Claim, ghg^{-1} has the same cycle decomposition.

Thm Two elements h, h' in S_n are conjugate (i.e. $\exists g$ such that $h' = ghg^{-1}$) iff they are products of disjoint cycles of the same length.
Pf: Use the conjugacy formula above to define g. Is g unique?

Cor The number of conjugacy classes in S_n = the number of disjoint cycle decompositions = the number of partitions of n.

ex: for S_5

e	$1+1+1+1+1$
(ab)	$2+1+1+1$
(abc)	$3+1+1$
$(abcd)$,	$4+1$
$(abcde)$	5
$(ab)(cd)$	$2+2+1$
$(abc)(de)$	$3+2$

Aside $n! \sim \sqrt{2\pi n}\, \frac{n^n}{e^n}$ $\quad \log n! \sim n\log n - n$
$p(n) = \#$ of partitions of $n \sim \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{2n/3}}$ $\quad \log p(n) \sim \sqrt{n}$

So the number of conjugacy classes in $S_{100} = p(100) = 190{,}569{,}292 \sim 10^9$
order of $S_{100} = 100! \sim 9 \times 10^{157}$
$\Rightarrow S_{100}$ a <u>very</u> non-abelian group.